JHEP HYPER VERSION

Received: *November 20, 1998,* Accepted: *December 1, 1998*

Spectral covers, charged matter and bundle cohomology

Duiliu-Emanuel Diaconescu

School of Natural Sciences
Institute for Advanced Study
Olden Lane, Princeton, NJ 08540
E-mail: diacones@sns.ias.ed

Gheorghe Ionesei

Department of Mathematics
University "Alexandru Ioan Cuza"
Iasi, 6600 Romania
E-mail: giones@uaic.ro

Abstract: We consider four dimensional heterotic compactifications on smooth elliptic Calabi-Yau threefolds. Using spectral cover techniques, we study bundle cohomology groups corresponding to charged matter multiplets. The analysis shows that in generic situations, the resulting charged matter spectrum is stable under deformations of the vector bundle.

Keywords: Superstring Vacua, F-Theory, Differential and Algebraic Geometry.

JHEP12(1998)001

Contents

1. Introduction

In the recent years, there has been important progress in understanding the mathematical structure of four dimensional $N = 1$ string vacua. Most of the new developments have been made possible by the discovery of F-theory [1, 2] and it's relation to heterotic models. At technical level, the duality relates two sets of seemingly very different geometric data. Namely, the heterotic string vacua are generally characterized by a complex holomorphic vector bundle V over a d-dimensional Calabi-Yau variety Z. On the other hand, the F-theory models are determined by an elliptically fibered $(d + 1)$-dimensional Calabi-Yau variety X. The problem of mapping the geometric moduli of the pair (V, Z) to those of X is very complicated. A systematic approach is based on the powerful construction of holomorphic bundles on elliptic fibrations presented in [3, 4, 5]. Various aspects of this map in different dimensions and including certain nonperturbative aspects have been discussed in [3, 4, 6, 7, 8, 9, 10, 11, 12, 13].

Although the string dynamics as well as the duality map are better understood in eight and six dimensions, phenomenologically interesting models eventually involve four dimensional compactifications. The latter present a series of peculiar aspects, qualitatively different from their higher dimensional counterparts. The present work is focused on the detailed understanding of one of these aspects, namely the charged matter spectrum localized along a codimension one locus Σ in the base of the elliptic fibration. In four dimensional models, under suitable conditions, Σ is an algebraic curve. These charged multiplets can be described from dual points of view either as zero modes associated to certain bundle cohomology groups or as degrees of freedom localized at the intersections of F-theory seven-branes. Here we take the first point of view which has the advantage of a precise description of the twisting line bundle on Σ.

Our analysis is focused on charged matter multiplets corresponding to cohomology groups of the form $H^1(Z,V)$ which can be localized according to [3]. We prove that the cohomology $H^1(Z,V)$ reduces to the cohomology $H^0(\Sigma,\mathcal{F})$ of a twisting line bundle $\mathcal{F}$ on Σ and we provide a detailed derivation of $\mathcal{F}$ using direct image techniques and the Grothendieck-Riemann-Roch theorem. Similar issues have been briefly considered in [12]. The precise computation of the cohomology groups depends on the particular aspects of the theory, but it can be performed explicitly in large classes of models.

The techniques developed here also allow us to address another question of interest. In general, we expect a nontrivial variation of the cohomology group $H^1(Z,V)$ under deformation of the vector bundle V. This is most easily seen by noting that in many cases the line bundle $\mathcal{F}$ corresponds to a special divisor on Σ. Therefore there is a potential variation in the number of holomorphic sections $h^0(\Sigma,\mathcal{F})$ as Σ moves in it's linear system on B. In physical terms, this means that the number of generation anti-generation pairs could vary function of the vector bundle moduli. We are able to prove that in generic situations i.e. for nonsingular reduced and irreducible spectral covers, the cohomology is stable under deformations of the vector bundle. It should be noted that this analysis does not exclude potential exceptional behavior which will be investigated elsewhere. Also, the approach is purely geometrical, therefore valid in perturbative string theory. Although it would be very interesting to understand how the results are affected by the full string dynamics, a complete treatment seems out of reach at the present stage.

The paper is structured as follows. Section 2 contains a brief review of the spectral cover construction and a detailed treatment of localization of cohomology. In section 3, we exploit the results of section 2 in order to determine the bundle cohomology and it's variation in a large class of models.

2. Spectral covers, cohomology and localized matter

The purpose of this section is twofold. We start with a brief review of the spectral cover construction of heterotic vector bundles following [3], mainly aimed at fixing notations and conventions. We then provide a more detailed treatment of localization of cohomology, clarifying and extending the present discussions in the literature.

The models considered in this paper are four dimensional $E_8\times E_8$ heterotic vacua with compactification data (Z,V_1,V_2). Here $\pi: Z\to B$ is a smooth elliptically fibered Calabi-Yau threefold with a section $\sigma: B\to Z$ over a rational surface B. Typically, B is isomorphic to a Hirzebruch surface F_e or to a del Pezzo surface dP_k. V_1,V_2 are two holomorphic bundles with structure groups $SU(n_{1,2})\subset E_8$ and $c_1(V_{1,2})=0$. In the following, Z will be taken to be a smooth Weierstrass model

$$zy^2=x^3-axz^2-bz^3 \tag{2.1}$$

in $\mathbf{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ with $\mathcal{L} \simeq K_B^{-1}$ in order to satisfy the Calabi-Yau condition. a, b are sections of $\mathcal{L}^4, \mathcal{L}^6$.

We concentrate on a single $SU(n)$ bundle V embedded in one E_8 factor. Since $c_1(V) = 0$, the restriction of V to a generic smooth fiber E_b, $b \in B$ must be flat, therefore $V|_{E_b}$ decomposes as a sum of flat holomorphic line bundles over E_b. These bundles can be described [3] in terms of spectral data $(C, \mathcal{N})$ where $C \subset B$ is a ramified n-fold cover of B and $\mathcal{N}$ is a line bundle on C. The spectral surface C is determined by the equation

$$a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2} = 0 \tag{2.2}$$

if n is even, the last term being $a_n x^{(n-3)/2} y$ for n odd. The coefficients a_i are sections of $\mathcal{M} \otimes \mathcal{L}^{-i}$ with no common zeroes, where $\mathcal{M}$ is a twisting line bundle pulled back from the base. Since the base is rational, $\mathcal{M}$ is uniquely determined by the class $\eta = c_1(\mathcal{M})$. It follows that C belongs to the linear system $\mathcal{O}_Z(n\sigma) \otimes \pi^*\mathcal{M}$. The bundle V is given by

$$V = p_{Z*}\left(p_C^*\mathcal{N} \otimes \mathcal{P}|_{Z\times_B C}\right), \tag{2.3}$$

where $p_Z : Z \times_B C \to Z$ and $p_C : Z \times_B C \to C$ are the natural projections and $\mathcal{P} = \mathcal{O}_{Z\times_B Z}(\Delta - \sigma_1 - \sigma_2) \otimes \mathcal{L}^{-1}$ is the Poincaré line bundle on $Z \times_B Z$. Note that the condition $c_1(V) = 0$ is equivalent to

$$c_1(\mathcal{N}) = \frac{1}{2}\left(K_C - \pi_C^* K_B\right) + \gamma, \tag{2.4}$$

where γ is a $(1,1)$ class on C in the kernel of[1] $\pi_C : C \to B$ [3]. The class γ can be chosen trivial, if the line bundle $K_C - \pi_C^* K_B$ has a square root on C. Since $K_C = \pi_C^*\eta + n\sigma|_C$, in general this holds if

$$\eta \equiv K_B \pmod 2, \qquad n \equiv 0 \pmod 2. \tag{2.5}$$

As explained in [3], this is the case if the bundle V is symmetric i.e. $\tau^* V \simeq V^\vee$ where $\tau : Z \to Z$ is the involution of the elliptic fibration. If the conditions 2.5 are not satisfied, γ has to be chosen of the form [3]

$$\gamma = \lambda\left(n\sigma - \pi_C^*\eta - n\pi_C^* K_B\right), \tag{2.6}$$

where λ is a suitable half-integral number.

[1] We use a distinct notation π_C for the restriction of $\pi : Z \to B$ to C in order to distinguish classes pulled back to C from classes pulled back to the threefold Z. This distinction will be important further on.

JHEP12(1998)001

2.1 Localized charged matter

In the above models, the charged matter content is determined by the bundle cohomology groups of the form $H^1(Z, R_i)$ where R_i are associated vector bundles on Z defined by the representations R_i of $SU(n)$. Throughout this paper we will exclusively consider matter multiplets associated to $H^1(Z, V)$ (the fundamental representation). The other multiplets are also interesting, but the present techniques do not provide an equally good control on the relevant cohomology spaces. Also, the following considerations are restricted to bundles arising from nonsingular reduced and irreducible spectral covers C. The behavior of cohomology in more general situations will be studied elsewhere.

As explained in [3, 12], the cohomology $H^1(Z, V)$ can be calculated using the Leray spectral sequence. This leads to the following exact sequence

$$0 \to H^1\left(B, R^0\pi_*V\right) \to H^1(Z, V) \to H^0\left(B, R^1\pi_*V\right) \to H^2\left(B, R^0\pi_*V\right). \tag{2.7}$$

Therefore the computation of $H^1(Z, V)$ reduces to the computation of the direct images $R^i\pi_*V$, $i = 0, 1, 2$. Since this involves certain subtleties, it will be presented in detail in the following.

First, note that since the fibers of π are one-dimensional, the fiber cohomology groups $H^2(Z_b, V_b)$, $b \in B$ vanish everywhere on B. In particular, $R^2\pi_*V \simeq 0$ is locally free and the base change theorem [14, Theorem III.12.11], shows that we have an isomorphism

$$\phi^1(b) : R^1\pi_*V \otimes k(b) \to H^1(Z_b, V_b), \tag{2.8}$$

for any $b \in B$. As observed in [3], the $H^1(Z_b, V_b)$ is non-zero if and only if b lies in the codimension one locus $\Sigma \subset B$ defined by

$$a_n = 0. \tag{2.9}$$

Since a_n is a section of the line bundle $\mathcal{M} \otimes {K_B}^n$, this is in fact a divisor in the linear system $|\eta + nK_B|$. Throughout this paper, we will assume that the generic divisor in $|\eta + nK_B|$ is a nonsingular irreducible holomorphic curve. This can be achieved, for example, if $\mathcal{M}$ is sufficiently ample. Therefore, the sheaf $R^1\pi_*V$ is a rank zero coherent sheaf supported on Σ. More precisely, $R^1\pi_*V$ can be represented as $i_*\mathcal{F}$ where $i : \Sigma \to B$ denotes the standard embedding and $\mathcal{F}$ is a rank one coherent sheaf on Σ. Note that $R^1\pi_*V \simeq i_*\mathcal{F}$ is not a locally free sheaf on B, therefore the base change theorem shows that the natural map

$$\phi^0(b) : R^0\pi_*V \otimes k(b) \to H^0(Z_b, V_b) \tag{2.10}$$

cannot be surjective. In order to obtain more information, we can use the results of [15, 16, 17] to show that $R^0\pi_*V$ vanishes. More precisely, since $\pi : Z \to B$ is a flat

JHEP12(1998)001

morphism, the natural map 2.10 is injective. We sketch the proof[2] which is formally identical to that of Proposition 2.7. of [15]. Let m_b denote the ideal sheaf of the point b on B. Since the morphism $\pi : Z \to B$ is flat, the $\pi^* m_b$ is the ideal sheaf of Z_b on Z, therefore we have an exact sequence

$$0 \to \pi^* m_b \to \mathcal{O}_Z \to \mathcal{O}_{Z_b} \to 0\,. \tag{2.11}$$

By tensoring with V, we obtain

$$0 \to \pi^* m_b \otimes V \to V \to V_b \to 0\,, \tag{2.12}$$

where V_b is extended by zero to Z. This yields by taking direct images

$$0 \to m_b \otimes R^0\pi_* V \to R^0\pi_* V \to R^0\pi_* V_b \simeq H^0\left(Z_b, V_b\right), \tag{2.13}$$

showing that

$$\phi^0(b) : R^0\pi_* V_b \otimes k(b) \to H^0\left(Z_b, V_b\right) \tag{2.14}$$

is injective.

Since we have also noted that $\phi^0(b)$ cannot be surjective and $\dim H^0\left(Z_b, V_b\right) = 1$, it follows that $R^0\pi_* V = 0$. Taking into account 2.7, this shows that

$$H^1(Z, V) \simeq H^0\left(B, R^1\pi_* V\right) \simeq H^0\left(\Sigma, \mathcal{F}\right). \tag{2.15}$$

Therefore, in order to complete the computation, we have to determine the rank one sheaf $\mathcal{F}$. A very useful technical result [15, 16] states that, since the fibers of $\pi : Z \to B$ are one-dimensional, the first image commutes with base change. Concretely, given a diagram of the form

$$\begin{array}{ccc} Z \times_B B' & \xrightarrow{p_Z} & Z \\ p_{B'}\big\downarrow & & \pi\big\downarrow \\ B' & \xrightarrow{u} & B \end{array}\,, \tag{2.16}$$

we have

$$u^* R^1\pi_* V \simeq R^1 p_{B'*}\left(p_Z^* V\right). \tag{2.17}$$

In general, this relation is not true unless $u : B' \to B$ is a flat base extension [14, Proposition III.9.3]. The fact that it holds in this case is a consequence of the base change theorem[3]. Applying this result for the diagram

$$\begin{array}{ccc} Z \times_B \Sigma & \xrightarrow{q} & Z \\ p_\Sigma\big\downarrow & & \pi\big\downarrow \\ \Sigma & \xrightarrow{i} & B \end{array}\,, \tag{2.18}$$

[2] A more direct argument, due to P. Aspinwall and E. Witten is based on the definition of the sheaf $R^0\pi_* V$. Just note that a section of $R^0\pi_* V$ over a Zariski open set in B vanishes almost everywhere since Σ is closed.

[3] We thank D. Hernández Ruipérez for helpful explanations on these issues.

JHEP12(1998)001

it follows that

$$\mathcal{F} \simeq i^* R^1 \pi_* V \simeq R^1 p_{\Sigma *} (q^* V) \,. \tag{2.19}$$

Now consider the diagrams

$$\begin{array}{ccccc} Z \times_B \Sigma & \xrightarrow{r} & Z \times_B C & \xrightarrow{p_Z} & Z \\ {\scriptstyle p_\Sigma}\Big\downarrow & & {\scriptstyle p_C}\Big\downarrow & & {\scriptstyle \pi}\Big\downarrow \\ \Sigma & \xrightarrow{j} & C & \xrightarrow{\pi_C} & B \end{array} \,. \tag{2.20}$$

Note that $q = p_Z \circ r$ and $i = \pi_C \circ j$ since $\Sigma = \sigma \cap C$ in Z. Applying the same technique, we obtain

$$R^1 p_{\Sigma *} (q^* V) \simeq R^1 p_{\Sigma *} (r^* p_Z^* V) \simeq j^* R^1 p_{C*} (p_Z^* V) \,. \tag{2.21}$$

The last piece of the puzzle is then provided by the relation between the vector bundle V on Z and the line bundle $\mathcal{N}$ on C. According to [15, 16, 17], the two objects are related by a Fourier-Mukai transform

$$\begin{aligned} V &\simeq R^0 p_{Z*} (p_C^* \mathcal{N} \otimes \mathcal{P}) \\ \mathcal{N} &\simeq R^1 p_{C*} \left(p_Z^* V \otimes \mathcal{P}^{-1} \otimes p_C^* \pi_C^* K_B^{-1} \right) . \end{aligned} \tag{2.22}$$

Using again the base change in 2.20, we find that

$$j^* \mathcal{N} \simeq R^1 p_{\Sigma *} \left(p_Z^* V \otimes p_\Sigma^* j^* \pi_C^* K_B^{-1} \right) , \tag{2.23}$$

since the restriction of $\mathcal{P}^{-1}$ to $Z \times_B \Sigma \subset Z \times_B \sigma$ is trivial. Therefore, the final result is

$$\mathcal{F} \simeq R^1 p_{\Sigma_*} (p_Z^* V) \simeq j^* (\mathcal{N} \otimes \pi_C^* K_B) \,, \tag{2.24}$$

which can be rewritten

$$\mathcal{F} \simeq j^* \mathcal{N} \otimes i^* K_B \,. \tag{2.25}$$

Note that $\mathcal{F}$ is a line bundle on Σ whose degree can be easily computed from 2.4 and 2.6

$$\deg \mathcal{F} = \frac{1}{2} \Sigma (\Sigma + K_B) + \lambda (\Sigma \eta) \,. \tag{2.26}$$

The above derivation can also be applied to the dual bundle $V^\vee$, leading to an interesting localized interpretation of chirality also noted in [12]. Relative duality for the finite flat morphism $p_Z : Z \times_B C \to Z$ ([5, Proposition 5.20]) yields

$$V^\vee \simeq R^0 p_{Z*} \left(p_C^* \mathcal{N}^{-1} \otimes \mathcal{P}^{-1} \otimes \omega_{Z \times_B C/Z} \right) , \tag{2.27}$$

where $\omega_{Z \times_B C/Z}$ is the relative dualizing sheaf. Since $K_Z \simeq \mathcal{O}_Z$, the latter is determined by

$$\omega_{Z \times_B C/Z} \simeq \omega_{Z \times_B C} \simeq p_C^* \left(K_C \otimes \pi_C^* K_B^{-1} \right) . \tag{2.28}$$

Therefore we obtain

$$V^\vee = R^0 p_{Z*}\left(p_C^*\left(\mathcal{N}^{-1}\otimes K_C\otimes \pi_C^* K_B^{-1}\right)\otimes \mathcal{P}^{-1}\right). \tag{2.29}$$

The inverse of 2.29 is given by

$$\mathcal{N}^{-1}\otimes K_C\otimes \pi_C^* K_B^{-1} = R^1 p_{C*}\left(p_Z^* V^\vee \otimes \mathcal{P}\otimes p_C^*\pi_C^* K_B^{-1}\right). \tag{2.30}$$

Repeating the steps 2.22-2.25, we find

$$R^1\pi_* V^\vee \simeq i_*\mathcal{G} \tag{2.31}$$

with

$$\mathcal{G}\simeq j^*\left(\mathcal{N}^{-1}\otimes K_C\right). \tag{2.32}$$

Comparing 2.25 and 2.32, it follows that

$$\mathcal{G}\simeq K_\Sigma\otimes \mathcal{F}^{-1}, \tag{2.33}$$

where

$$K_\Sigma \simeq j^* K_C\otimes i^* K_B, \tag{2.34}$$

since $\Sigma = C\cap\sigma$ in the Calabi-Yau threefold Z. This yields the promised localization of chirality

$$\begin{aligned} h^1(Z,V) - h^1(Z,V^\vee) &= h^0(\Sigma,\mathcal{F}) - h^0\left(\Sigma, K_\Sigma\otimes\mathcal{F}^{-1}\right) = \\ &= \deg\mathcal{F} - g(\Sigma) + 1 = -\lambda(\Sigma\eta), \end{aligned} \tag{2.35}$$

by Riemann-Roch on Σ. Note that the result agrees with the computation of $c_3(V)$ in [12].

Since the above derivation is quite abstract, it is instructive to perform an independent check of the result by applying Grothendieck-Riemann-Roch theorem for the morphism $\pi : Z\to B$. Although this does not completely determine the direct images, it provides significant information on their structure. Following [18], we have

$$\mathrm{ch}(\pi_! V)\,\mathrm{Td}(B) = \pi_*\left(\mathrm{ch}(V)\mathrm{Td}(Z)\right). \tag{2.36}$$

The left hand side of the above equation can be computed taking into account ([3, 12]) that

$$\begin{aligned} \mathrm{Td}(Z) &= 1 + \frac{1}{12}\pi^*\left(c_2(B) + 11 c_1(B)^2\right) + \sigma\pi^* c_1(B), \\ \mathrm{ch}(V) &= 1 - \sigma\pi^*\eta - \pi^*\omega + \lambda\eta(\eta - n c_1(B)) w_Z, \end{aligned} \tag{2.37}$$

where ω is a class on the base and w_Z is the fundamental class of Z. Since π_* annihilates all classes pulled back from the base, we are left with

$$\begin{aligned} \pi_*\left(\mathrm{ch}(V)\mathrm{Td}(Z)\right) &= -\eta - nK_B + \lambda\eta\left(\eta + nK_B\right) w_B = \\ &= -\Sigma + \lambda\left(\eta\Sigma\right) w_B, \end{aligned} \tag{2.38}$$

where w_B is the fundamental class of B and the intersection $(\eta\Sigma)$ is computed on B. On the other hand,

$$\mathrm{ch}\left(\pi_! V\right) = \mathrm{ch}\left(R^0\pi_* V\right) - \mathrm{ch}\left(R^1\pi_* V\right). \tag{2.39}$$

According to the above analysis, $R^0\pi_* V$ vanishes and $R^1\pi_* \simeq i_*\mathcal{F}$ where $\mathcal{F}$ is a line bundle on Σ determined by 2.25. Applying the Grothendieck-Riemann-Roch theorem for the immersion $i : \Sigma \to B$ [18], we find

$$\mathrm{ch}\left(R^1\pi_* V\right)\mathrm{Td}(B) = i_*\left(\mathrm{ch}(\mathcal{F})\mathrm{Td}\left(\Sigma\right)\right). \tag{2.40}$$

The right hand side of the above equation can be easily evaluated obtaining

$$\Sigma + \left(\deg(\mathcal{F}) - \frac{1}{2}\deg\left(K_\Sigma\right)\right) w_B\,. \tag{2.41}$$

Using 2.4, 2.25 and 2.34, we are finally left with

$$\Sigma - \lambda\left(\eta\Sigma\right) w_B\,, \tag{2.42}$$

which is in precise agreement with 2.38.

To summarize, we have showed that the computation of $H^1(Z,V)$ reduces by localization to the computation of the line bundle cohomology group $H^0\left(\Sigma,\mathcal{F}\right)$, where Σ is a non-singular irreducible curve. As detailed in the next section, this simplification allows an explicit evaluation in a large number of cases. Moreover, this description can be used to answer another problem of interest, namely the dependence of the cohomology groups on the vector bundle moduli.

3. Bundle cohomology and variation

As a starting point, let us consider the vector bundle moduli in the spectral cover realization. According to [3, 5], the moduli of the bundle V can be associated to either deformations of the spectral surface C or of the line bundle $\mathcal{N} \to C$. If the twisting line bundle $\mathcal{M}$ is sufficiently ample, $H^1\left(C, \mathcal{O}_C\right) = 0$ and $\mathcal{N}$ has no deformations. [3, 5]. Assuming that this is the case, we first concentrate on the variation of cohomology under deformations of the spectral surface C. The line bundle $\mathcal{N}$ is kept fixed and determined by it's first Chern class 2.4. At a latter stage we will show that, if this condition is relaxed, the results are not affected by deformations of $\mathcal{N}$.

We first consider the τ-invariant case, when the conditions 2.5 are satisfied and λ can be set to zero. It will be shortly argued that this is in fact the most mathematically interesting case. Then, the spectral line bundle $\mathcal{N}$ is uniquely determined by

$$\mathcal{N} = \left(\pi_C^*\mathcal{M} \otimes \mathcal{O}_Z(n\sigma)|_C \otimes \pi_C^* K_B^{-1}\right)^{1/2}\,, \tag{3.1}$$

where the square root exists by 2.5. Note that these conditions actually imply a stronger statement, namely that the line bundle $\pi^*(\mathcal{M} \otimes K_B^{-1}) \otimes \mathcal{O}_Z(n\sigma)$ admits a square root on Z. This leads to a better understanding of the problem of variation of cohomology as follows.

The line bundle $\mathcal{F}$ determined in 2.24 is isomorphic to the restriction of the line bundle

$$\left(\pi^* \left(\mathcal{M} \otimes K_B\right) \otimes \mathcal{O}_Z(\sigma)\right)^{1/2} \tag{3.2}$$

from Z to Σ. Since Σ is included in $\sigma \simeq B$, we can first restrict to σ, obtaining

$$\mathcal{F} \simeq i^* \left(\mathcal{M} \otimes K_B^{(n+1)}\right)^{1/2}, \tag{3.3}$$

where in the right hand side we have now the restriction of a *fixed* line bundle $\mathcal{T} = \left(\mathcal{M} \otimes K_B^{(n+1)}\right)^{1/2}$ on B. Therefore the initial problem reduces to studying the dependence of the cohomology group

$$H^0\left(\Sigma, i^* \left(\mathcal{M} \otimes K_B^{(n+1)}\right)^{1/2}\right) \tag{3.4}$$

on the moduli of C. In generic situations, the spectral surface C moves in the linear system $|n\sigma + \pi^*\eta|$ on Z [4, 5]. The number of parameters is $h^0(C, K_C)$ and can be computed using the Riemann-Roch theorem. Alternatively, the deformations of C can be associated to variations of the sections $a_0, a_2 \ldots a_n$ of $\mathcal{M}, \mathcal{M}\otimes\mathcal{L}^{-2} \ldots \mathcal{M}\otimes\mathcal{L}^{-n}$. Since Σ is defined by $a_n = 0$, it is clear that varying $a_0, a_2 \ldots a_{n-1}$ leaves 3.4 unchanged. Therefore, the relevant moduli correspond to deformations of $a_n \in H^0\left(B, \mathcal{M} \otimes K_B^n\right)$ which move Σ in the linear system $|\eta + nK_B|$. It follows that the variation of the cohomology of V reduces to the variation of 3.4 when Σ moves $|\eta + nK_B|$, keeping the line bundle $\left(\mathcal{M} \otimes K_B^{(n+1)}\right)^{1/2}$ fixed on B. Note that 2.26 and the adjunction formula give

$$\deg\left(\mathcal{F}\right) = g(\Sigma) - 1\,. \tag{3.5}$$

Therefore the isomorphism class of $\mathcal{F}$ defines a point in the Jacobian variety $J_{g(\Sigma)-1}(\Sigma)$ of degree $g(\Sigma) - 1$ of Σ. The problem is then equivalent to studying the motion of the corresponding point in $J_{g(\Sigma)-1}(\Sigma)$ when Σ moves in the linear system $|\eta + nK_B|$. Potential variations in the cohomology can in principle occur since the divisor class associated to $\mathcal{F}$ is special on Σ [19, 20, 21]. The number of holomorphic sections can jump if the class of $\mathcal{F}$ crosses the singular strata $W^r_{g(\Sigma)-1}$ of the Θ-divisor.

If we consider non-invariant cases, when $\lambda \neq 0$, it can be easily inferred from 2.26 that the class of $\mathcal{F}$ is no longer special. Therefore, the number of holomorphic section is determined by Riemann-Roch theorem and is independent of deformations of Σ. Since the discussion has been so far rather general, we consider next certain concrete situations.

3.1 Concrete Computation

Here we consider models with the base B isomorphic to a Hirzebruch surface $B \simeq F_e$. In order to avoid certain technical complications, we consider $e \geq 2$. The conclusions are also valid for $e = 0, 1$ but the analysis has to be modified on a case by case basis. Let C_0, f denote the standard generators of the Picard group of B, satisfying $C_0^2 = -e$, $f^2 = 0$, $C_0 \cdot f = 1$. Note that the canonical class of B is given by

$$K_B = -2C_0 - (e+2)f\,. \tag{3.6}$$

We pick the class of Σ to be of the form

$$[\Sigma] = aC_0 + bf, \qquad a, b \in Z_+\,. \tag{3.7}$$

According to [14, Cor. V.2.18], the linear system aC_0+bf contains an irreducible nonsingular curve different from C_0, f if and only if $a > 0$ and $b \geq ae$. The conditions 2.5 are satisfied if

$$a \equiv 0 \ (\mathrm{mod}\ 2), \qquad b \equiv e \ (\mathrm{mod}\ 2) \tag{3.8}$$

and n is even. Note that this implies $a \geq 2$, $b \geq 2e$.

The class of the fixed line bundle $\mathcal{T}$ in 3.3 is of the form

$$[\mathcal{T}] = \frac{1}{2}\left((a-2)C_0 + (b-(e+2))f\right). \tag{3.9}$$

Taking into account the above restrictions on a, b, the generic divisor in the above linear system is an irreducible nonsingular curve Γ if $a > 2$. Hence, the line bundle $\mathcal{T}$ can be taken of the form $\mathcal{O}_B(\Gamma)$ for fixed Γ. If $a = 2$, it follows from 3.9 that Γ can be taken a collection of disjoint P^1 fibers of B. Our problem is then to compute $H^0(B, \mathcal{O}_\Sigma(\Gamma))$ and to understand it's eventual variation. Consider the exact sequence[4]

$$0 \to \mathcal{O}_B(\Gamma - \Sigma) \to \mathcal{O}_B(\Gamma) \to \mathcal{O}_\Sigma(\Gamma) \to 0\,. \tag{3.10}$$

The associated long exact cohomology sequence yields

$$\begin{aligned} 0 \to H^0(B, \mathcal{O}_B(\Gamma)) \to H^0(\Sigma, \mathcal{O}_\Sigma(\Gamma)) \to \\ \to H^1(B, \mathcal{O}_B(\Gamma - \Sigma)) \to H^1(B, \mathcal{O}_B(\Gamma)) \to \ldots \end{aligned} \tag{3.11}$$

Now consider the exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{O}_B(\Gamma) \to \mathcal{O}_\Gamma(\Gamma) \to 0\,. \tag{3.12}$$

The last term is the normal bundle of Γ, $N_{\Gamma/B}$, extended by zero to B. Since $H^i(B, \mathcal{O}_B) = 0$ for $i > 0$, the long exact cohomology sequence yields

$$0 \to H^1(B, \mathcal{O}_B(\Gamma)) \to H^1\left(\Gamma, N_{\Gamma/B}\right) \to 0\,. \tag{3.13}$$

[4]We thank I. Dolgachev for explaining this line of argument to us.

Next, we have

$$\deg\left(N_{\Gamma/B}\right) = \Gamma^2$$
$$2g(\Gamma) - 2 = \Gamma(\Gamma + K_B)\,. \tag{3.14}$$

A direct computation shows that if $a > 2$, $\Gamma \cdot K_B < 0$, hence

$$2g(\Gamma) - 2 < \deg\left(N_{\Gamma/B}\right) \tag{3.15}$$

and the divisor $N_{\Gamma/B}$ is not special on Γ. Therefore we obtain

$$H^1(B, \mathcal{O}_B(\Gamma)) \simeq H^1\left(\Gamma, N_{\Gamma/B}\right) \simeq 0\,. \tag{3.16}$$

If $a = 2$, Γ is a sum of disjoint rational fibers whose normal bundles are trivial, therefore 3.16 is still valid. Moreover, since

$$[\Gamma] = \frac{1}{2}([\Sigma] + K_B)\,, \tag{3.17}$$

Kodaira-Serre duality on B shows that

$$H^1(B, \mathcal{O}_B(\Gamma - \Sigma)) \simeq H^1(B, \mathcal{O}_B(\Gamma))^\vee \simeq 0\,. \tag{3.18}$$

Consequently, 3.11 reduces to

$$0 \to H^0(B, \mathcal{O}_B(\Gamma)) \to H^0(\Sigma, \mathcal{O}_\Sigma(\Gamma)) \to 0\,. \tag{3.19}$$

This shows that the cohomology group $H^0(\Sigma, \mathcal{O}_\Sigma(\Gamma))$ does not vary as Σ moves in it's linear system. Furthermore, it can be explicitly computed as the cohomology of a fixed line bundle on the base, which follows from Riemann-Roch theorem. To this end, note that

$$H^2(B, \mathcal{O}_B(\Gamma)) \simeq H^0(B, \mathcal{O}_B(K_B - \Gamma)) \simeq 0\,. \tag{3.20}$$

Since we have also proved that the first cohomology group vanishes, the Riemann-Roch formula yields

$$h^0(\mathcal{O}_B(\Gamma)) = \frac{1}{2}\Gamma(\Gamma - K_B) + 1 = \frac{1}{8}a(2b - ea)\,. \tag{3.21}$$

Collecting all the results of this section, we have shown that in a large class of models, the cohomology groups $H^1(Z, V)$ can be precisely computed function of spectral data and they are stable under deformations of the vector bundle. Similar methods can be applied in other cases as well, leading to similar conclusions (for example when the base B is a rational elliptic surface). Inspecting the above chain of arguments suggests that this is the generic behavior in elliptic models over rational surfaces when the class η is sufficiently ample.

As mentioned in the beginning of the section, the analysis has been so far restricted to spectral surfaces C with $H^1(C, \mathcal{O}_C) = 0$. In the following, we show that the conclusions are still valid when this condition is relaxed.

3.2 Nontrivial $H^1(C, \mathcal{O}_C)$

According to [5, Lemma 5.16], $H^1(C, \mathcal{O}_C)$ can be computed starting from the exact sequence

$$0 \to \mathcal{O}_Z(-n\sigma) \otimes \pi^* \mathcal{M}^{-1} \to \mathcal{O}_Z \to \mathcal{O}_C \to 0 \tag{3.22}$$

which yields

$$\begin{aligned} \ldots \to H^1(Z, \mathcal{O}_Z) \to H^1(C, \mathcal{O}_C) \to \\ \to H^2\left(Z, \mathcal{O}_Z(-n\sigma) \otimes \pi^* \mathcal{M}^{-1}\right) \to H^2(Z, \mathcal{O}_Z) \to \ldots \end{aligned} \tag{3.23}$$

Since in our case Z is a Calabi-Yau variety, $H^1(Z, \mathcal{O}_Z) \simeq H^2(Z, \mathcal{O}_Z) \simeq 0$, therefore we obtain an isomorphism

$$H^1(C, \mathcal{O}_C) \simeq H^2\left(Z, \mathcal{O}_Z(-n\sigma) \otimes \pi^* \mathcal{M}^{-1}\right). \tag{3.24}$$

The right hand side can be evaluated using the Leray spectral sequence

$$\begin{aligned} H^1(C, \mathcal{O}_C) \simeq H^1\left(B, \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}\right) \oplus \\ \oplus\ H^1\left(B, \mathcal{L} \otimes \mathcal{M}^{-1}\right) \oplus \ldots \oplus H^1\left(B, \mathcal{L}^{n-1} \otimes \mathcal{M}^{-1}\right). \end{aligned} \tag{3.25}$$

Now consider the exact sequence

$$0 \to \mathcal{O}_C(-\Sigma) \to \mathcal{O}_C \to \mathcal{O}_\Sigma \to 0\,. \tag{3.26}$$

The associated long exact cohomology sequence reads

$$\begin{aligned} 0 \to H^0(C, \mathcal{O}_C) \xrightarrow{r} H^0(\Sigma, \mathcal{O}_\Sigma) \to H^1(C, \mathcal{O}_C(-\Sigma)) \\ \to H^1(C, \mathcal{O}_C) \xrightarrow{f} H^1(\Sigma, \mathcal{O}_\Sigma) \to \ldots \end{aligned} \tag{3.27}$$

The map r is clearly surjective since any constant function on Σ can be extended on C, therefore the sequence truncates as

$$0 \to H^1(C, \mathcal{O}_C(-\Sigma)) \to H^1(C, \mathcal{O}_C) \xrightarrow{f} H^1(\Sigma, \mathcal{O}_\Sigma) \to \ldots \tag{3.28}$$

By tensoring the exact sequence 3.22 by $\mathcal{O}_Z(-\sigma)$, we obtain

$$0 \to \mathcal{O}_Z(-(n+1)\sigma) \otimes \pi^* \mathcal{M}^{-1} \to \mathcal{O}_Z(-\sigma) \to \mathcal{O}_C(-\Sigma) \to 0\,. \tag{3.29}$$

Note that we can use the Calabi-Yau property of Z and the rationality of σ to show that $H^1(Z, \mathcal{O}_Z(-\sigma)) \simeq 0$ and $H^2(Z, \mathcal{O}_Z(-\sigma)) \simeq 0$. This follows from the long cohomology exact sequence associated to

$$0 \to \mathcal{O}_Z(-\sigma) \to \mathcal{O}_Z \to \mathcal{O}_\sigma \to 0\,. \tag{3.30}$$

JHEP12(1998)001

Therefore we can repeat the steps 3.22-3.25 to derive

$$H^1(C, \mathcal{O}_C(-\Sigma)) \simeq H^1(C, \mathcal{O}_C) \oplus H^1\left(B, \mathcal{L}^n \otimes \mathcal{M}^{-1}\right). \tag{3.31}$$

However, the line bundle $\mathcal{L}^n \otimes \mathcal{M}^{-1}$ is isomorphic to $\mathcal{O}_B(-\Sigma)$ and we have that $H^1(B, \mathcal{O}_B(-\Sigma)) \simeq 0$ by an exact sequence argument similar to those presented so far. Therefore, the complex vector spaces $H^1(C, \mathcal{O}_C(-\Sigma))$ and $H^1(C, \mathcal{O}_C)$ are isomorphic, implying that the linear map f maps $H^1(C, \mathcal{O}_C)$ to zero in 3.28. This shows that deformations of the line bundle $\mathcal{N}$ on C are mapped to trivial deformations of the restriction $\mathcal{N}|\Sigma$. Hence they have no effect on the cohomology, as claimed before.

Acknowledgments

We are very grateful to Paul Aspinwall, Gottfried Curio, Igor Dolgachev, Daniel Hernández Ruipérez, Liviu Nicolaescu, Govindan Rajesh and Edward Witten for very helpful discussions and suggestions.

References

[1] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[2] D.R. Morrison and Cumrun Vafa, *Compactifications of F-Theory on Calabi–Yau Threefolds – I*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114]; *Compactifications of F-theory on Calabi–Yau threefolds – II*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161].

[3] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Comm. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[4] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-theory*, *Nucl. Phys.* **B 505** (1997) 165 [hep-th/9701165].

[5] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9709029.

[6] G. Curio and D. Lust, *A class of $N = 1$ dual string pairs and its modular superpotential*, *Int. J. Mod. Phys.* **A 12** (1997) 5847 [hep-th/9703007].

[7] B. Andreas, G. Curio and D. Lust, $N = 1$ *dual string pairs and their massless spectra*, *Nucl. Phys.* **B 507** (1997) 175 [hep-th/9705174].

[8] G. Curio, *Chiral multiplets in $N = 1$ dual string pairs*, *Phys. Lett.* **B 409** (1997) 185 [hep-th/9705197].

[9] B. Andreas and G. Curio, *Three-branes and five-branes in N=1 dual string pairs*, *Phys. Lett.* **B 417** (1998) 41 [hep-th/9706093].

[10] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, *JHEP* **04** (1998) 019 [hep-th/9802194].

[11] G. Curio and R.Y. Donagi, *Moduli in $N = 1$ heterotic/F-theory duality*, *Nucl. Phys.* **B 518** (1998) 603 [hep-th/9801057].

[12] G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett.* **B 435** (1998) 39 [hep-th/9803224].

[13] B. Andreas, *On vector bundles and chiral matter in $N = 1$ heterotic compactifications*, hep-th/9802202.

[14] R. Hartshorne, *Algebraic geometry* Springer-Verlag, New York 1993.

[15] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez and J.M. Muñoz Porras, *Mirror symmetry on K3 surfaces via Fourier-Mukai transform*, *Comm. Math. Phys.* **195** (1998) 79 [alg-geom/9704023].

[16] D. Hernández Ruipérez and J.M. Muñoz Porras, *Structure of the moduli of stable sheaves on elliptic fibrations*, alg-geom/9809019.

[17] P.S. Aspinwall and R.Y. Donagi, *The heterotic string, the tangent bundle and derived categories*, hep-th/9806094.

[18] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin Heidelberg New-York 1984.

[19] P. Griffiths and J. Harris, *Principles of algebraic geometry* Wiley-Interscience, New-York 1978.

[20] P. Griffiths, *An introduction to the theory of special divisors on algebraic curves*, Regional Conference Series in Mathematics no 44, American Mathematical Society, Providence, Rhode Island 1980.

[21] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of algebraic curves*, vol. 1, Springer-Verlag, New-York 1985.